Lecture 2

06-09-2018

Recall: Last time we have considered the equation

$$y'(t) = p(t)\,y(t) + g(t)$$

We use the method of integrating factor

$\mu(t) = e^{-\int p(t)\,dt}$ and solve $y(t)$ as

$$\boxed{y(t) = e^{\int p(t)\,dt}\left[\int e^{-\int p(t)\,dt} g(t)\,dt + C\right]}$$

§ Existence and uniqueness theorem:

Thm: (Linear case)

Let $I = (a,b)$ with the IVP

$$\begin{cases} y'(t) = p(t)\,y(t) + g(t). \\ y(t_0) = y_0 \end{cases}$$

Assume: $p(t)$, $g(t)$ continuous on (a,b)

then: $\exists!$ solution $y(t)$ to the IVP.

Pf: By method of integrating factor :

$$\mu(t) = e^{\int -p(t)\,dt}$$

and

$$y(t) = \frac{1}{\mu(t)} \left[\int_{t_0}^{t} \mu(s) g(s)\,ds + c \right]$$

is general solution to the equation on (a,b).

Uniqueness :

if we have two c_1, c_2 s.t. they both satisfy

IVP $\Rightarrow$ $\frac{c_1}{\mu(t_0)} = \frac{c_2}{\mu(t_0)} \Rightarrow c_1 = c_2$.

Existence : Given $t_0 \in (a,b)$, y_0.

Want to solve

$$y(t_0) = y_0.$$

i.e. $\frac{1}{\mu(t_0)} [F(t_0) + c] = y_0. \Rightarrow c = \mu(t_0) y_0 - F(t_0).$

where $F'(t) = g(t)\mu(t)$.

If we $F(t) = \int_{t_0}^{t} g(s)\mu(s)\,ds \Rightarrow F(t_0) = 0$

And we have $y(t) = \frac{1}{\mu(t)} \left[\int_{t_0}^{t} \mu(s) g(s)\,ds + \mu(t_0) y_0 \right]$

is the unique solution to the IVP.

Thm (General existence & uniqueness for 1^{st} order ODE)

Consider on (a,b) with the IVP:

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

where $f: \underset{t}{(a,b)} \times \underset{y}{(c,d)} \longrightarrow \mathbb{R}$ with

f and $\frac{\partial f}{\partial y}$ continuous in $(a,b)\times(c,d)$

Then 1) (Existence) there exist $\delta>0$ with $(t_0-\delta, t_0+\delta) \subseteq (a,b)$ such the IVP has a solution $y(t)$ in $(t_0-\delta, t_0+\delta)$

2) (Uniqueness) Any two solution $y_1(t), y_2(t)$ with $y_i: (a,b) \longrightarrow (c,d)$ to the IVP are equal.

Rk: Existence is local while uniqueness is global.

Pf: (defered to later section).

Example: consider $\begin{cases} ty' + 2y = 4t^2 \\ y(1) = 2 \end{cases}$

find the domain s.t. the existence and uniqueness holds.

First for $t \neq 0$, we can rewrite the equation as

$$y' = \underbrace{\frac{-2}{t}}_{p(t)} y + \underbrace{4t}_{q(t)} \quad \text{which is linear.}$$

- $p(t)$, $q(t)$ continuous on $(0, +\infty)$

$\Rightarrow$ Exist an unique solution $y(t)$ to IVP on $(0, +\infty)$

In fact we can solve the linear equation

$$e^{\int p(t)\,dt} = e^{-2\log|t|} = \frac{1}{t^2}$$

$$\int e^{\int p(t)\,dt} q(t)\,dt = \int 4t^3\,dt = t^4$$

Hence we have $y(t) = \frac{1}{t^2}(t^4 + c)$

For the IVP : $y(t) = \frac{1}{t^2}(t^4 + 1)$

which cannot be extended through $t = 0$

$\Rightarrow$ maximal domain $= (0, +\infty)$

- If we change the initial value to $y(1) = 1$

$\Rightarrow$ $y(t) = t^2$ which is globally defined

Example: (non-uniqueness example)

$$y' = y^{1/3}, \quad y(0) = 0$$

$$f(t,y) = y^{1/3}, \quad \frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3}$$ Not cont. at $y=0$.

Solve: $$\frac{y'}{y^{1/3}} = 1 \Rightarrow \frac{3}{2} y^{2/3} = t + c$$

IV: $y(0) = 0 \Rightarrow c = 0$

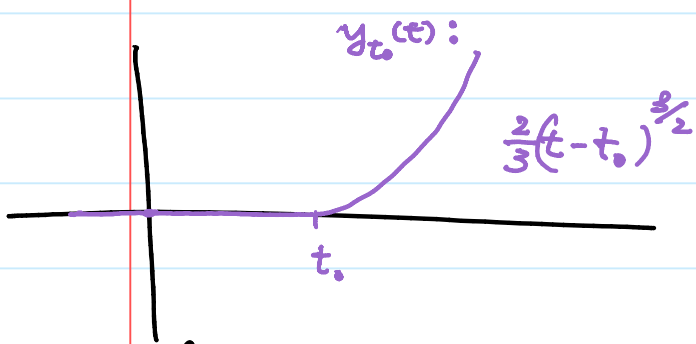
$\therefore \quad y = \left(\frac{2}{3}|t|\right)^{3/2}$ is a solution

On the other hand: $y \equiv 0$ is also a sol.

Hence we do NOT have the uniqueness

Moreover, we let

$$y_{t_0}(t) := \begin{cases} 0 & 0 \le t \le t_0 \\ \left(\frac{2}{3}(t-t_0)\right)^{3/2} & t_0 \le t \end{cases}$$



each $y_{t_0}(t)$ is a solution to the IVP

- We even have infinite family of solution $y_{t_0}(t)$
- It does NOT contradict uniqueness because $\frac{\partial f}{\partial y}$ is NOT continuous at $t=0$

Example: (non global existence)

$$\begin{cases} \dfrac{dy}{dt} = y^2 \quad = f(t,y) \\ y(0) = 1. \end{cases}$$

$$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dt} = 1 \quad \Rightarrow \quad \frac{-1}{y} = t + c$$

IV $\Rightarrow$ $C = -1$

Hence we have $y = \dfrac{-1}{t-1}$ as solution

Notice that $\lim_{t \to 1} y(t) = \pm\infty$ so there is no way to extend the solution through $t = 1$.

Example: (uniqueness application)

$$\begin{cases} y' = \sin(e^t y) = f(t,y). \\ y(0) = 0 \end{cases}$$

$$f(t,y) = \sin(e^t y), \quad \frac{\partial f}{\partial y}(t,y) = e^t \cos(e^t y)$$

both are continuous.

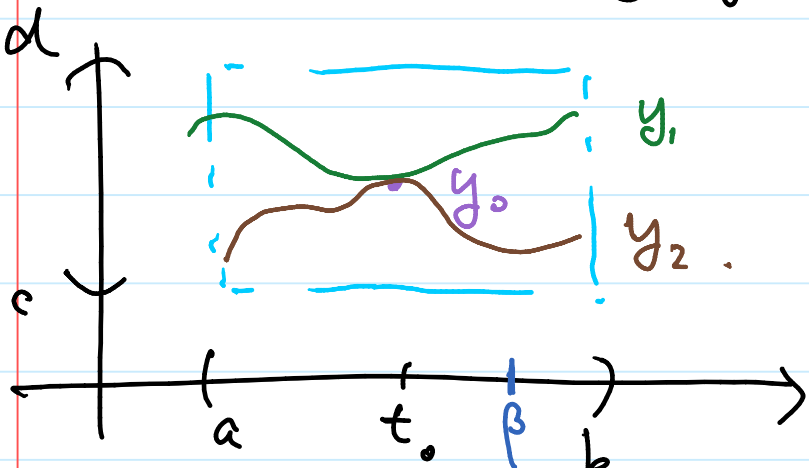
(in fact f is smooth function on $\mathbb{R}^2$).

Observe: $y \equiv 0$ is the unique solution to the IVP by the uniqueness theorem.

Rk: In the book, the uniqueness result is only in $(t_0-\delta, t_0+\delta)$ which is local.

But in fact we may prove the global uniqueness as follows:

Assume: $f: (a,b)\times(c,d) \longrightarrow \mathbb{R}$, with two solution y_1, y_2 to IVP $f(t_0)=y_0$



We let

$\beta := \sup\{ s \mid t_0 \le s < b$ with $y_1(t)=y_2(t)$ in $[t_0, s)\}$.

Want: $\beta = b$

First, local uniqueness $\Rightarrow$ $t_0 < \beta$

and we assume $\beta < b$.

we have $y_1(t) = y_2(t)$ in $[t_0, \beta)$

$y_1(\beta) = y_2(\beta) = y_{\beta,0}$ by continuity

Local uniqueness again for the IVP $y(\beta) = y_{\beta,0}$

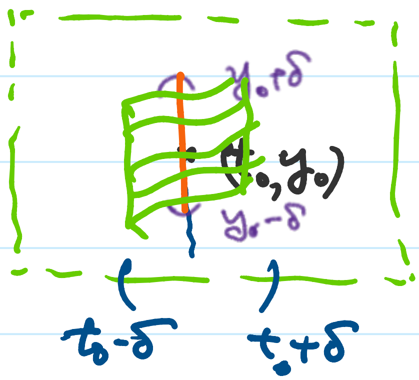
$\Rightarrow$ $\exists\, \delta$ s.t. $y_1 = y_2$ in $(\beta-\delta, \beta+\delta)$

which leads to contradiction.

§ C^1-dependence on initial condition

Let $f: (a,b) \times (c,d) \longrightarrow \mathbb{R}$ s.t.

- $f, \frac{\partial f}{\partial y}$ is continuous as in the Existence and uniqueness theorem



- $\exists$ δ small (possible smaller than we originally have in the existence and uniqueness theorem).

<u>s.t.</u> the IVP $\begin{cases} y' = f(t,y) \\ y(t_0) = z \in (y_0-\delta, y_0+\delta) \end{cases}$

has unique solution $y_z(t)$ for $t \in (t_0-\delta, t_0+\delta)$

<u>Furthemore</u>: if we put together the unique solution

$$Y : (t_0-\delta, t_0+\delta) \times (y_0-\delta, y_0+\delta)$$

by $Y(t,z) := y_z(t)$

<u>Conclusion</u>: Y is C^1 in t and y.